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RESEARCH ARTICLE

Restrained Steiner Number of Some Special Graphs

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ABSTRACT

A set W of vertices of a graph G is a restrained Steiner set if W is a Steiner set, and if either W=V or the subgraph G[V-W] induced by V-W has no isolated vertices. The minimum cardinality of a restrained Steiner set of G is the restrained Steiner number of G, and is denoted by $s_r(G)$. This article explores the restrained Steiner number of some special graphs.

Keywords: Steiner set, Steiner number, Isolated vertices, Restrained Steiner number, Special graphs.

1. INTRODUCTION

Finite graphs are considered with neither loops nor multiple edges. For any graph G, the vertex set is denoted by V(G) and the edge set by E(G). The order of G is denoted by p = |V(G)| and the size by q = |E(G)|. For every vertex, v϶V, the open neighborhood N(v) is the set {u ϵ V/uv ϵ E} and the closed neighborhood of v is the set N[v] = N(v) U{v}. The distance d(u,v) between two vertices u and v in a connected graph G is the length of a shortest u–v path in G. A u-v path of length d(u,v) is called a u–v geodesic. It is known that this distance is a metric on the vertex set V(G). Basic graph theoretic terminology is referred from [1-3]. For a vertex v of G, the eccentricity e(v) is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, rad G and the maximum eccentricity is its diameter, diam G of G. Two vertices x and y are antipodal if d(x, y) = diam G.

A vertex v is an extreme vertex or a simplical vertex of a graph G if the subgraph induced by its neighbors is complete [4]. Note that every end vertex is simplical. If e = uv is an edge of a graph G with d(u) = 1 and d(v) > 1, then we call e a pendant edge, u a leaf or end vertex and v a support. For a nonempty set W of vertices in a connected graph G, the Steiner distance d(W) of W is the minimum size of a connected subgraph of G containing W. Necessarily, each subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W-tree. S(w) denotes the set of all vertices that lie on Steiner W-trees. A set W ⊑ V(G) is called a Steiner set of G if every vertex of G lies on some Steiner W-tree or if S(W) = V(G). A Steiner set of minimum cardinality is a minimum Steiner set or simply a s-set and this cardinality is the Steiner number s(G) of G [4].

The restrained concept on geodetic sets is introduced in [5, 6]. A set W of vertices of a graph G is a restrained Steiner set if W is a Steiner set, and if either W = V or the subgraph G(V-W) induced by V-W has no isolated vertices. [7, 8]. For the graph G given in figure 1, W = {v_1, v_3} is the unique minimum Steiner set of G and so s(G) = 2. The three Steiner W–trees are shown below. Since the subgraph G(V-W) has an isolated vertex v4, W is a not a restrained Steiner set of G. If W = {v1, v3, v4}, then W is connected and the Steiner W–tree contains vertices of W only.

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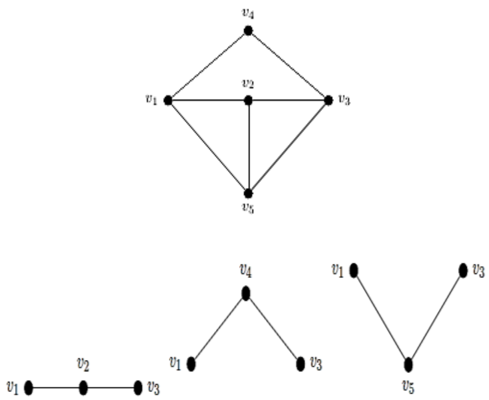

Figure 1.Steiner W-trees

Therefore $W = \{v_1, v_2, v_3, v_4, v_5\}$ is the unique restrained Steiner set of G. Hence $s_r(G) = 5$. Each restrained Steiner set is a Steiner set, and the converse is not true.

2. PRELIMINARY NOTES

This section includes some definitions and their results are used in the subsequent section.

Definition 2.1 [9] The (m, n) tadpole graph denoted by $T_{m,n}$ consists of a cycle graph on m (atleast 3) vertices and a path graph on n vertices, connected with a bridge. It is also known as a dragon graph or a kite.

Definition 2.2 The square of a path (cycle) is the graph obtained by joining every pair of vertices of distance two in the path (cycle).

Definition 2.3 The corona of two graphs G_1 and G_2 is the graph $G = G_1 \Theta G_2$ formed by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_1.

Definition 2.4 The graph obtained by joining a single pendant edge to each vertex of a path is called a comb. i. e., a comb is the corona of P_n and K_1.

Definition 2.5 The triangular snake T_n is obtained from the path P_n by replacing every edge of a path by a triangle C_3.

Definition 2.6 The double triangular snake DT_n is obtained from a path P_n with vertices $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$ and to a new vertex u_i for $i = 1, 2, \ldots, n-1$.

Definition 2.7 An alternate triangular snake $A(T_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternately) to a new vertex v_i, that is every alternate edge of a path is replaced by C_3.

Definition 2.8 A double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes which have a common path.

Definition 2.9 [2] The wheel graph $W_n = K_1 + C_n$ is the combination of K_1 and C_n with the order $p = n + 1$. The vertex of K_1 is known as the apex vertex.

Definition 2.10 [9] The helm graph H_n is the graph obtained from the wheel graph W_n by adding a pendant edge to each vertex in the rim of W_n. Thus a helm graph has $2n + 1$ vertices.

Definition 2.11 The flower graph Fl_n is obtained from the helm graph H_n by joining each pendant vertex to the apex of H_n.

Definition 2.12 The sun flower graph SFl_n is obtained from the flower graph Fl_n by adding n pendant vertices and joining them to the apex of the flower graph Fl_n.

Theorem 2.13 [4], [7] Each extreme vertex of a graph G belongs to every Steiner and restrained Steiner set of G.

Theorem 2.14 [7] There is no graph of order p with $s_r(G) = p - 1$.

3. RESTRAINED STEINER NUMBER OF SOME SPECIAL GRAPHS

Theorem 3.1 For a tadpole graph $G = T_{m,n} (m \geq 3)$, $s_r(G)$ is defined as in (3.1).

$$s_r(G) = \begin{cases} 2 & \text{if m is even} \\ 3 & \text{if m is odd} \end{cases} \quad (3.1)$$

Proof Let $G = T_{m,n}$ be the tadpole graph obtained from the path $P_n (n \geq 2)$ and the cycle $C_m (m \geq 3)$ connected by a bridge. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n and $u_1, u_2, \ldots, u_m$ be the vertices of the cycle C_m. Without loss of generality, we assume that v_n and u_1 are adjacent. The graph $G = T_{m,n}$ is shown in figure 2.

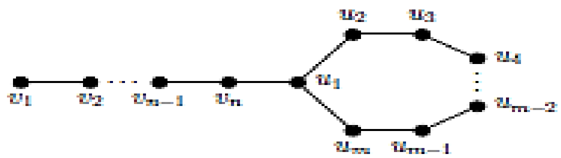

Figure 2.Tadpole graph $T_{m,n}$

The restrained Steiner number of $G = T_{m,n}$ depends on the number of vertices of the cycle C_m.

Case 1 m is even Let W be a Steiner set of G. By theorem 2.13, $v_1 \in W$. Let $w \in C_m$ be the antipodal vertex of v_1. Then $W = \{v_1, w\}$ is a Steiner set of G. Since the subgraph G(V - W) has no isolated vertices, W is a restrained Steiner set of G, so that $s_r(G) = 2$.

Case 2 m is odd Let W be a Steiner set of G. It is lucid that $v_1 \in W$. Since m is odd, let $y, z \in C_m$ be the two antipodal vertices for the vertex v_1. Then $W = \{v_1, y, z\}$ is a restrained Steiner set of G so that $s_r(G)=3$.

Theorem 3.2 For a triangular snake $G = T_n$, $(n \geq 4)$, $s_r(G) = n + 1$.

Proof Let $G = T_n$ be a triangular snake obtained from the path P_n: $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex u_i for i=1, 2,…., n-1. The graph $G = T_n$ is shown in figure 3.

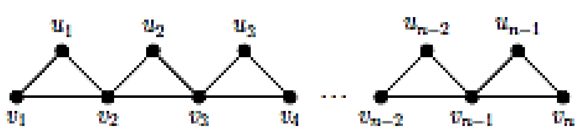

Figure 3.Triangular snake T_n

Let $W = \{v_1, v_n, u_1, u_2, \ldots, u_{n-1}\}$ be the set of all extreme vertices of G. By theorem 2.13, W is a subset of every restrained Steiner set of G. Since S(W) = V(G) and the subgraph G(V-W) has no isolated vertices, W is a restrained Steiner set of G. Therefore $s_r(G) = |W| = n - 1 + 2 = n + 1$.

Theorem 3.3 For the comb $G = P_n \odot K_1 (n \geq 2)$, $s_r(G) = n$.

Proof Let P_n: $v_1, v_2, \ldots, v_n$ be a path on n vertices. Let $u_1, u_2, \ldots, u_n$ be the pendant vertices attached to $v_1, v_2, \ldots, v_n$ respectively. The resulting graph is the comb $G = P_n \odot K_1$ as shown in figure 4.

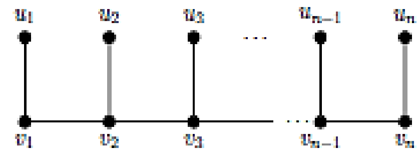

Figure 4.Comb $G = P_n \odot K_1$

Let $W=\{u_1, u_2, \ldots, u_n\}$ be the set of all end vertices of G. Since each end vertex is an extreme vertex, by theorem 2.13, W is a subset of every Steiner set of G. Since the subgraph G(V - W) has no isolated vertices, W is a restrained Steiner set of G. Hence $s_r(G) = n$.

Theorem 3.4 For a square path $G = P_n^2 (n \geq 4)$, $s_r(G)$ is defined as in (3.2).

$$s_r(G) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd} \end{cases} \quad (3.2)$$

Proof Let the vertices of the square path $G = P_n^2$ be $v_1, v_2, \ldots, v_n$. The graph $G = P_n^2$ is shown in figure 5.

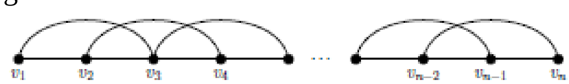

Figure 5.Square path P_n^2

Case 1 n is even The set $W = \{v_1, v_n\}$ is the unique minimum Steiner set of G, so that every vertex of g lies on Steiner W-trees. Since the subgraph G(V - W) has no isolated vertices, W is the unique restrained Steiner set of G and so $s_r(G) = |W| = 2$.

Case 2 n is odd It is observed that no two element subset of $G = P_n^2$ is a Steiner set of G and so $s_r(G) \geq 3$. Let $W_1 = \{v_1, v_{n-1}, v_n\}$ and $W_2 = \{v_1, v_2, v_n\}$, then $W_i (i = 1, 2)$ is the only two minimum Steiner sets of G. Since the subgraphs $G[V-W_i]$ $(i = 1, 2)$ have no isolated vertices, $W_i (i = 1, 2)$ is a restrained Steiner set of G, so that $s_r(G) = 3$.

Theorem 3.5 For a double triangular snake, $G = DT_n$, $s_r(G) = 2n$.

Proof Let G = DT_n be the double triangular snake obtained from the path P_n: $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex W_i for i=1, 2,…., n-1 and to a new vertex u_i for i = 1, 2,…., n-1. The graph is shown in figure 6.

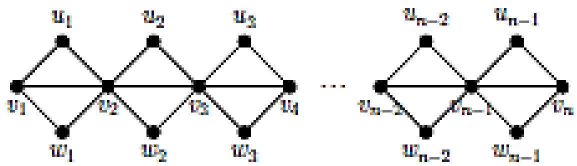

Figure 6.Double triangular snake DT_n

Let W = $\{u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}\}$ be the set of all extreme vertices of G. By theorem 2.13, W is a subset of every Steiner set of G, and so $s(G) \geq 2(n - 1) = 2n - 2$. Since the vertices v_1 and v_n do not lie on the Steiner W-tree, W is not a Steiner set of G. Let W' = W ∪ $\{v_1, v_n\}$. Then W' is a Steiner set of G. Since the subgraph G(V - W′) has no isolated vertices, W' is a restrained Steiner set of G. Therefore $s_r(G) = |W'| = 2n + 2 = 2n$.

Theorem 3.6 For an alternate triangular snake G = $A(T_n)$ ($n \geq 4$), $s_r(G)$ is defined as in (3.3).

$$s_r(G) = \begin{cases} \left[\frac{n-1}{2}\right] + 2 & \text{if n is odd} \\ \left[\frac{n}{2}\right] + 2 & \text{if n is even, triangle starts at } v_1 \\ \left[\frac{n-1}{2}\right] + 2 & \text{if n is even, triangle starts at } v_2 \end{cases} \quad (3.3)$$

Proof Let P: $v_1, v_2, \ldots, v_n$ be a path of order n. The alternate triangular snake G = $A(T_n)$ is obtained by replacing every alternate edge of the path P by a triangle. We consider two cases.

Case 1 n is odd Let the vertices of G be $v_1, v_2, \ldots, v_{n-1}, u_1, u_2, \ldots, u_{\frac{n-1}{2}}$. Since n is odd, either v_1 or v_n is an end vertex of G. Without loss of generality we assume that the triangle starts at v_1 and so v_n is the end vertex of G. The alternate triangular snake G = $A(T_n)$ when n is odd is shown in figure 7(a).

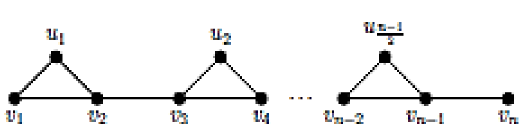

Figure 7(a).Alternate triangular snake $A(T_n)$

Let W be a Steiner set of G. Let U = $\{v_1, v_n, u_1, u_2, \ldots, u_{\frac{n-1}{2}}\}$be the set of all extreme vertices of G. By theorem 2.13, U⊑W. Therefore $s(G) \geq \left[\frac{n-1}{2}\right] + 2$. Since S(U) = V(G), U is a Steiner set of G. Since the subgraph G(V-W) has no isolated vertices, U is a restrained Steiner set of G so that $s_r(G) = |U| = \left[\frac{n-1}{2}\right] + 2$.

Case 2 n is even When n is even, we come across two cases depending on the edges replaced by the triangles.

Subcase 2(a) Triangle starts at v_1 Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n/2}$ be the vertices of G. The graph G is shown in figure 7(b)

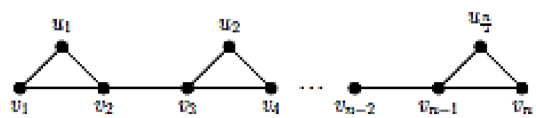

Figure 7(b).Alternate triangular snake G

Let W be a Steiner set of G. Let U = $v_1, v_n, u_1, u_2, \ldots, u_{n/2}$ g be the set of all extreme vertices of G. By theorem 2.13, U⊑W. Therefore $s(G) \geq [n / 2] + 2$. Since S(U) = V(G), U is a Steiner set of G. Since the subgraph G(V - U) has no isolated vertices, U is a restrained Steiner set of G, so that $s_r(G) = |U| = [n / 2] + 2$.

Subcase 2(b) Triangle starts at v_2 Let the vertices of the alternate triangular snake be $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\frac{n-1}{2}}$. Graph G is shown in figure 7(c).

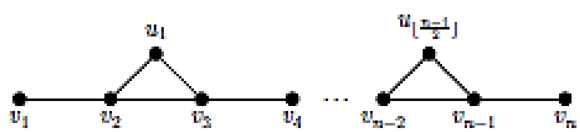

Figure 7(c).Alternate triangular snake G

Clearly each $u_i(1 \leq i \leq \left[\frac{n-1}{2}\right])$ is an extreme vertex of G. Also the vertices v_1 and v_n are extreme vertices of G. Let W = $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{[n-1/2]}\}$. Since S(W) = V(G), W is a Steiner set of G. It is clear that the subgraph G(V-W) has no isolated vertices. Therefore W is a restrained Steiner set of G, so that $s_r(G) = |W| = \left[\frac{n-1}{2}\right] + 2$.

Theorem 3.7 For a double alternate triangular snake $G = DA(T_n)$ $(n \geq 4)$, $s_r(G)$ is defined as in (3.4).

$$S_r(G) = \begin{cases} n+1 & \text{if n is odd} \\ n+2 & \text{if n is even, triangle starts at } v_1 \\ 2\left[\frac{n-1}{2}\right] + 2 & \text{if n is even, triangle starts at } v_2 \end{cases} \qquad (3.4)$$

Proof Let P: $v_1, v_2, \ldots, v_n$ be a path of order n. The double alternate triangular snake $G = DA(T_n)$ is obtained by replacing both sides of every alternate edge of the path P by a triangle. We consider two cases.

Case 1 n is odd Let the vertices of $G = DA(T_n)$ be $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\frac{n-1}{2}}, w_1, w_2, \ldots, w_{\frac{n-1}{2}}$. Without the loss of generality, we assume that the triangles start at v_1 and so v_n is the end vertex of G. The double alternate triangular snake $G = DA(T_n)$ when n is odd is shown in figure 8(a).

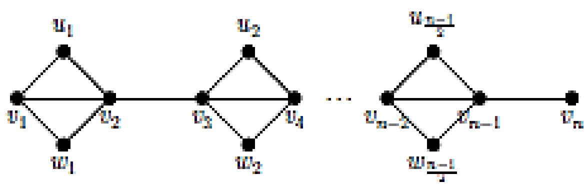

Figure 8(a).Double alternate triangular snake $DA(T_n)$

Let W be a Steiner set of G. Let $U = v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\frac{n-1}{2}}, w_1, w_2, \ldots, w_{n-1/2}$ be the set of all extreme vertices of G. By theorem 2.13, $U \sqsubseteq W$. Therefore $s(G) \geq 2\left[\frac{n-1}{2}\right] + 2$. Since $S(U) = V(G)$, U is a Steiner set of G. Since the subgraph G(V-U) has no isolated vertices, U is a restrained Steiner set of G, so that $s_r(G) = |U| = 2\left[\frac{n-1}{2}\right] + 2 = n+1$.

Case 2 n is even When n is even, we come across two cases depending on the edges replaced by the triangles.

Subcase 2(a) Triangles start at v_1 Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n/2}, w_1, w_2, \ldots, w_{n/2}$ be the vertices of $G = DA(T_n)$. The graph G is shown in figure 8(b).

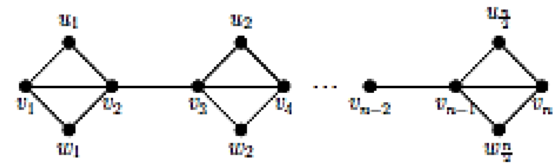

Figure 8(b).Double alternate triangular snake G

Let W be a Steiner set of G. Let $U = v_1, v_n, u_1, u_2, \ldots, u_{n/2}, w_1, w_2, \ldots, w_{n/2}$ be the set of all extreme vertices of G. By theorem 2.13, $U \sqsubseteq W$. Therefore $s(G) \geq 2[n/2] + 2$. Since $S(U) = V(G)$, U is a Steiner set of G. Since the subgraph G(V-U) has no isolated vertices, U is a restrained Steiner set of G, so that $s_r(G) = |U| = 2[n/2] + 2 = n + 2$.

Subcase 2(b) Triangles start at v_2 Let the vertices of the alternate triangular snake be $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\frac{n-1}{2}}, w_1, w_2, \ldots, w_{\frac{n-1}{2}}$. The graph G is shown in figure 8(c).

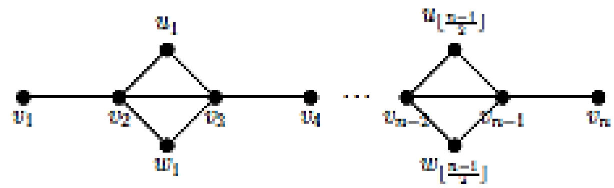

Figure 8(c).Double alternate triangular snake G

Clearly each u_i $(1 \leq i \leq \left[\frac{n-1}{2}\right])$ and w_i $(1 \leq i \leq \left[\frac{n-1}{2}\right])$ are extreme vertices of G. Also the vertices v_1 and v_n are extreme vertices of G. Let = $\{v_1, v_n, u_1, u_2, \ldots, u_{\frac{n-1}{2}}, w_1, w_2, \ldots, w_{\frac{n-1}{2}}\}$. Since $S(W) = V(G)$, W is a Steiner set of G. It is clear that the subgraph G(V-W) has no isolated vertices. Therefore W is a restrained Steiner set of G, so that $s_r(G) = |W| = 2\left[\frac{n-1}{2}\right] + 2$.

Theorem 3.8 For the helm graph $G = H_n$, $(n \geq 4)$, $s_r(G) = n + 1$.

Proof Let the vertices of the helm graph $G = H_n$ be $v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ where v is the apex vertex, $v_1, v_2, \ldots, v_n$ are the rim vertices of the wheel graph and $u_1, u_2, \ldots, u_n$ are the pendant vertices attached to $v_1, v_2, \ldots, v_n$ respectively. Let W be a Steiner set of G. Let $U = u_1, u_2, \ldots, u_n$ be the set of all extreme vertices in G. By theorem 2.13, $U \sqsubseteq W$ and so $s(G) \geq n$. Since the vertex v does not lie in the Steiner U–tree, U is not a Steiner set for G. Therefore the vertex v must belong to every S set of G. Thus W = v, u_1, u_2,…., u_n is the unique Steiner set of G. Thus s(G) = |W| = n + 1. Since the subgraph G(V-W) has no isolated vertices, W is a restrained Steiner set of G so that $s_r(G)$ = |W| = n + 1.

Theorem 3.9 For the flower graph G = Fl_n, (n ≥ 3), $s_r(G)$ = 2n + 1.

Proof Let v, v_1, v_2,…., v_n, u_1, u_2,…, u_n be the vertices of the flower graph G = Fl_n. Let U = {u_1, u_2,…, u_n } be the set of all extreme vertices of G. By theorem 2.13, U⊑W and so s(G) ≥ n. Since the vertices v_1, v_2,…., v_n do not lie on the Steiner U –tree, U is not a Steiner set of G. Let W' = U ∪ {v_1, v_2,…. v_n}W'= u ∪ {v_1, v_2,…., v_n}, then W' is a Steiner set of G and so s(G) = |W'| = 2n. Since p - 1 = 2n and the subgraph G(V-W') has an isolated vertex v, W' is not a restrained Steiner set of G. Thus W = W' ∪ {v} is the unique restrained Steiner set of G. Hence $s_r(G)$ = |W| = 2n + 1.

Corollary 3.10 For the sunflower graph G = SFl_n, (n ≥ 3), $s_r(G)$ = 3n + 1.

Proof Let v, v_1, v_2,…., v_n, u_1, u_2,…, u_n be the vertices of the flower graph Fl_n. Let w_1, w_2,…, w_n be the pendant vertices attached to the apex vertex v of the flower graph Fl_n and the graph G = SFl_n is obtained. It is obvious that $s_r(G)$ = $s_r(Fl_n)$ + n = (2n + 1) + n = 3n + 1.

4. CONCLUSION

Distance in graphs has many applications. Steiner tree problem is a distance related invariant one which is used in combinatorial optimization and computer science especially in design of computer circuits. They have numerous applications in industries. In this work, we have explored the restrained Steiner number of some special graphs. It would be interesting to find the restrainer Steiner number for other graphs also by various graph operations.

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